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A DIFFUSION APPROXIMATION ANALYSIS OF THE SUPPLY CORPS OFFICER MANPOWER SYSTEM

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THESIS

A DIFFUSION APPROXIMATION ANALYSIS OF THE SUPPLY CORPS OFFICER MANPOWER SYSTEM

by

Armando R. Solis

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A Diffusion Approximation Analysis of the Supply Corps Officer Manpower System

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ABSTRACT

A stochastic analysis of the Supply Corps officer manpower system is presented. The analysis is based on the concept of diffusion approximations. The state of a rank is represented as the superposition of a deterministic process, and a random noise variable, both of which are derived utilizing a set of formulated recursion equations. Numerical results are presented for a 3-rank system and compared against a simulation of a Markovian manpower model.

A Diffusion Approximation analysis is performed utilizing United States Navy Supply Corps data, in an effort to determine the probability of maintaining each rank, any two ranks, and/or the entire system between specified limits.

TABLE OF CONTENTS

I. INTRODUCTION

This paper attempts to examine manpower problems utilizing stochastic techniques. The point of departure is the deterministic theory of control given in Bartholomew [Ref. 1], and the object is to see how concepts and strategies originating in that context perform in a stochastic environment of the kind likely to be encountered in practice. Conclusions reached, both in this thesis and in Bartholomew [Ref. 3], suggest that, although useful, deterministic analysis is incomplete since it fails, for example, to provide required information concerning the variance of individual ranks. Without such information, manpower planners may attempt to "fine tune" the system by increasing or decreasing promotion or recruitment rates when in reality the fluctuations are due to the uncertainty in the system.

This study concentrates its efforts on one particular manpower system--that being the Supply Corps of the United States Navy. An effort is made to develop a stochastic model that provides the capability for simple computation. In addition, it will give the mean number of individuals in each rank, as do deterministic models; and as added features, will provide second order moments and distribution of force information. Armed with these key elements we can then attempt to answer questions related to the maintenance of desired force structures and with the ability to attain such structures.

The numbers of people in particular ranks change over time as a result of attrition, promotion and recruitment. Some of these flows are subject to management control while others vary due to a multitude of circumstances and may best be modelled stochastically. Factors such as the need to offer adequate career prospects, while at the same time, maintaining prescribed force levels are the manpower planners chief goals. A problem arises in that the planner must determine whether these goals can be achieved simultaneously, and, if so, how best to achieve them.

The manpower control problem has received a good deal of attention in recent years--see, for example, Bartholomew [Refs. 1,2,3], Davies [Refs. 4,5] and Grinold and Stanford [Ref. 6].

Although most of the work on manpower control has been based on stochastic models, the analysis has been deterministic in the sense that only expected values for attrition and promotion rates are used. More realistically, attrition and other flows do vary randomly and are therefore better represented by random variables. Recent work which does attempt to incorporate randomness into manpower control are Bartholomew [Refs. 2,3] and Kim [Ref. 7].

In this thesis we will assume that only promotion rates and recruitment rates are subject to control. While it may be true that attrition can be dramatically increased, utilizing techniques such as "Reductions in Force", it is the

problem of being unable to hold down attrition rates that plagues the present day Navy. As a result, the model developed in this thesis utilizes, as an input parameter, the attrition rate and is capable of determining, in a deterministic sense, the promotion and recruitment rates required to maintain the system at some specified level. Henceforth, the concept of being able to maintain a specified force structure will be referred to as maintainability; i.e., a structure is maintainable if and only if there are promotion and recruitment rates such that for each rank in the system, the expected number of individuals who depart during each time period is equal to the expected number of individuals who arrive during that same time period. The majority of this thesis will deal with the maintainability of desired force structures, but it turns out that this topic cannot be treated adequately without considering the question of whether those structures can be attained from present levels.

Since for all practical purposes, recruitment into the Supply Corps takes place only into the lowest rank, promotion control alone is left as a means of controlling the force distribution. Control by promotion has the disadvantage that fluctuations in the promotion rate are liable to create internal stresses within the organization. On the other hand, it was shown in Bartholomew [Ref. 1, Chapter 4], that in a deterministic environment, promotion control is capable of maintaining a much wider range of structures than recruitment control alone.

Having discussed the various areas of consideration, the general scope of this paper will be to present the following:

A. A basic deterministic Markovian model will be derived in Chapter II, to be used as a benchmark with which to compare the stochastic version.

B. Realizing that a deterministic representation is not representative of the random factors that affect most processes, a more realistic stochastic model will be derived in Chapters III and IV utilizing a Diffusion Approximation.

C. Results of a simulation of a stochastic Markovian manpower model will be presented in Chapter V and compared to results of the Diffusion Approximation as developed in this thesis. Shortfalls of the approximation will be discussed and advantages of the approximation pointed out.

D. A comparison of the deterministic and stochastic models will be made in Chapter VI. In particular, the correlation between adjacent ranks, and its effect on the ability to maintain desired force structures will be discussed.

E. Areas of possible further study will be briefly mentioned in Chapter VI. An effort is made to identify possible research topics, whose results, when used in conjunction with the model derived in this thesis, may reflect to an even greater degree the stochastic nature of a manpower system.

II. DETERMINISTIC ANALYSIS

A. A DETERMINISTIC MANPOWER MODEL FOR THE SUPPLY CORPS

Before one attempts to model any system, one must be familiar with its everyday operations. In an effort to compare the results of the approximation against a benchmark, we will use our familiarity of the system to make several educated assumptions as to the values of promotion and attrition rates within the Supply Corps. A compilation of these educated guesses, will define a Markovian transition matrix; the elements of which are the fractions of individuals in each rank that stay in the rank or are promoted in each time period. This matrix will then be compared against one where historical data is used as a basis for determining the elements of the Markovian transition matrix. Comparison between the two will answer the question of whether organizational assumptions made are reasonable.

The organizational assumptions that we will make are as follows:

1. We will assume that the fraction of Supply Corps officers with a particular number of years in a specified rank remains almost constant; that is, we will assume that the fraction of the Supply Corps force that are second year Lieutenants or third year Lieutenant Commanders, etc., remains the same.

2. Under present promotion patterns, promotion from Ensign (O-1) to Lieutenant Junior Grade (O-2) occurs at the

2 year point in an officer's career. In addition, promotion, for all practical purposes, is assured for all Ensigns completing their second year. It is therefore assumed in light of assumption 1, that approximately 50% of all O-1's are completing their second year and will therefore be promoted to O-2.

3. Similarly, it takes two years to transition between ranks O-2 and Lieutenant Commander (O-3); however, only 95% of those in the second year in rank O-2 get promoted. Therefore, this implies that 50% of all O-2's are in the promotion zone, and 95% of those eligible are promoted. This information yields the promotion fraction equal to 0.475 (0.50 x 0.95).

4. Using similar logic the following table values are computed:

	Years Required in grade prior to being in zone	Assumed percentage of total grade in zone	Assumed promotion rate for in zone	Promotion fraction of all members in grade
Ensign (O-1)	2	0.5000	1.0000	0.5000
Lieutenant Junior Grade (O-2)	2	0.5000	0.9500	0.4750
Lieutenant (O-3)	5	0.2000	0.9000	0.1800
Lieutenant Commander (O-4)	7	0.1429	0.8000	0.1143
Commander (O-5)	5	0.2000	0.6000	0.1200

In order to complete the transition matrix, it is necessary to determine reasonable attrition rates. Attrition rate data was obtained from historical data files at the Naval Military Personnel Command. Attrition rates cannot be forecasted with great accuracy, due in great measure to their correlation to many economic and social factors. Thus it appears reasonable to use a simple moving average as a predictor. We will use

$$\frac{V_i(t-1) + V_i(t)}{2}$$

as a forecasted value for the current attrition rate for stage i at time t+1.

This average appears to be a better forecasting technique than that of simply using the last attrition rate. A reason for not considering more terms in the attrition forecast, is the fact that we want the forecasted attrition rate to reflect only the recent past and not to consider forces which may have affected the manpower system at one time but are no longer present.

The remaining element required in the transition matrix is the continuation rate. Since the sum of the continuation rate, the attrition rate and the promotion rate must equal one, it is a simple matter to compute the remaining elements of the transition matrix. It should be mentioned once again, that these assumptions are not related to any historical data, except for attrition rates and existing "in zone" promotion rates. The resulting matrix is computed merely to compare against the matrix derived in the next subsection.

TRANSITION MATRIX

P_1 =

	O-1	O-2	O-3	O-4	O-5	O-6	Attrition rates "V"
O-1	0.4728	0.5	0	0	0	0	0.0272
O-2	0	0.3895	0.475	0	0	0	0.1355
O-3	0	0	0.7079	0.1800	0	0	0.1121
O-4	0	0	0	0.8467	0.1143	0	0.0390
O-5	0	0	0	0	0.8010	0.1200	0.0790
O-6		0	0	0	0	0.8378	0.1622

B. PROMOTION RATES REQUIRED TO MAINTAIN THE SUPPLY CORPS AUTHORIZED FORCE SIZE

The objective of this subsection, is to find out if the force levels which the Supply Corps has been authorized, are in fact maintainable. The authorized force levels mentioned are included in Appendix A, and are obtained from the Officer Programmed Authorizations, Fiscal Years 1979-1983 [Ref. 9]. In an effort to work with a single force level, the average force level was computed for the five years mentioned above; and it is this level that we attempt to maintain. It should be noted that over the 5 years under consideration, neither the individual ranks nor total force strength vary significantly, so the average force appears to be a reasonable force level to maintain.

If we determine that the desired force can be maintained, we will then have a transition matrix which can be compared against the matrix P_1 derived in subsection II.A. This will give us an idea of how drastically the system would have to change in order to maintain the desired force levels.

Before we proceed, a few definitions for the above:

$\tilde{Q}$ $\equiv$ a six component row vector whose elements are the number in each rank.

P $\equiv$ a 6 x 6 matrix whose elements contain promotion and continuation rates. (See matrix P_1, subsection II.A for the general form of this matrix.)

$\tilde{V}^T$ $\equiv$ a column vector whose elements contain the appropriate attrition rates for each rank.

$\underset{\sim}{R}$ ≡ the recruitment vector, which directs the flow of recruits into the proper ranks (fraction of recruits for each rank).

In the Supply Corps, the recruitment vector is not generally amenable to control, i.e., recruitment for the most part occurs into the lowest rank, i.e., R = [1,0,0,0,0,0]. In such cases we wish to know what degree of control can be exercised by the promotion rates. For a fixed size organization, what we wish to do is to find a transition matrix P, which allows us to maintain our desired structure Q. This implies that P must satisfy

$$\underset{\sim}{Q} = \underset{\sim}{Q} P + \underset{\sim}{Q}\,\underset{\sim}{V}^T \underset{\sim}{R} \qquad \text{(II.B.1)}$$

where $\underset{\sim}{Q}\,\underset{\sim}{V}^T \underset{\sim}{R}$ yields the number of recruits necessary to exactly equal the number of attritions at each time period. As was noted in subsection II.A, the matrix P is super-diagonal in form, i.e., non-zero elements only in the main diagonal and the diagonal above it. Bartholomew [Ref. 1] shows that when P has the super-diagonal form, equation (II.B.1) admits a unique solution which may be tested to see whether its elements are non-negative. If the matrix P is found to be non-negative, then the structure Q can be maintained.

As was mentioned previously, while we assume that recruitment takes place only into the lowest rank, we recognize the fact some recruitment does occur in ranks O-2 and O-3. However, the number recruited into these ranks is small

and hence will be neglected. Also, the attrition rates are as considered previously, i.e.,

$$V = [0.0272, 0.1355, 0.1121, 0.0390, 0.0790, 0.1622].$$

We note that equation (II.B.1) is equivalent to

$$\underset{\sim}{Q}\,P = \underset{\sim}{Q} - \underset{\sim}{Q}\,V^T \underset{\sim}{R}$$

and, at this point it becomes more convenient to abandon the matrix notation temporarily and to proceed with the components as does Bartholomew [Ref. 1]:

$$Q_1 P_{11} = Q_1 - R_1 \sum_{i=1}^{k} Q_i V_i$$

$$Q_1 P_{12} + Q_2 P_{22} = Q_2 - R_2 \sum_{i=1}^{k} Q_i V_i$$

$$\vdots$$

$$Q_4 P_{56} + Q_4 P_{66} = Q_6 - R_6 \sum_{i=1}^{k} Q_i V_i \;.$$

(The above are components of the respective vectors or matrices mentioned at the beginning of this subsection.)

Eliminating P_{ii} using $P_{ii} = 1 - V_i - P_{i,i+1}$ $(i = 1,2,\ldots, k-1)$ and using our special structure for the vector R yields the following equation:

$$P_{i,i+1} = \sum_{j=i+1}^{k} Q_j V_j / Q_i \qquad \text{(II.B.2)}$$

In other words, equation (II.B.2) says that the proportions required to be promoted from grade i must equal the number leaving from all grades above grade i divided by the size of grade i.

For example, given a point estimate for the attrition rate, and given that the promotion rates for O-6's is essentially zero; it is possible to compute the number of O-6's who depart the system. Given this value, one can then compute the number of O-5's who must be promoted each year to exactly maintain the required level of O-6's. The process continues in this fashion until we compute the required number of new accessions required to maintain the O-1 Rank. It is in this way that promotion rates can be calculated to exactly maintain the desired structure (assuming that we are at the structure we wish to maintain). If we are at some other force level and we wish to attain our goal, similar calculations can be done, but review must be done to ensure that the resulting promotion rates are within acceptable bounds as specified by organization policy.

For example, suppose we wish to maintain the following structure:

Total Force Level = 3955

Rank	Strength	Attrition Rates
O-1	621	0.0272
O-2	625	0.1355
O-3	995	0.1121
O-4	863	0.0390
O-5	617	0.0790
O-6	234	0.1622

We first compute the number of O-6's lost. This is equal to (234)(0.1622) = 37.9548. To maintain the desired structure 37.9548 O-5's on the average must be promoted. Therefore, the promotion rate required is (37.9548) $\div$ (617) = 0.0615. An important note to make, is that promotion rate is used in relation to the total rank strength, to determine what the rate would be in regards to "In the promotion zone" strength, one must first determine the number to be promoted, and then divide that number by the number "in the zone". In our example, promotion rate related to "In the zone" would be 37.9548 $\div$ (number of O-5's in zone). The other promotion rates can be similarly computed, and as a final result we obtain the matrix P_2.

Comparing this matrix with P_1 reveals that the reasonable guesses for the existing system are very close, in most cases, to the rates necessary for maintaining the desired system on the average. This implies that very little has to be done in

TRANSITION MATRIX

P_2 =

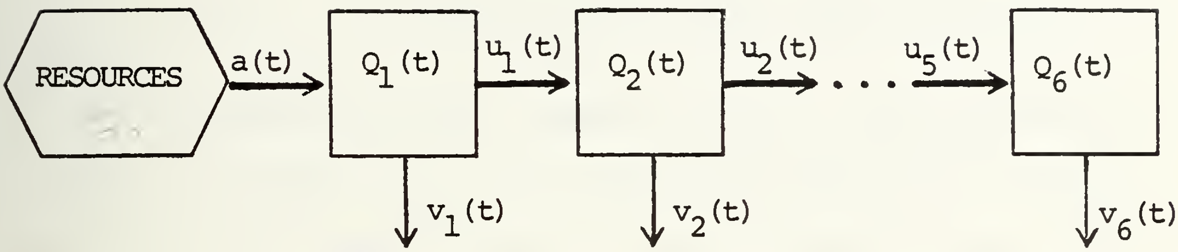
	O-1	O-2	O-3	O-4	O-5	O-6	"V"
O-1	.4629	.5099	0	0	0	0	.0272
O-2	0	.4934	.3711	0	0	0	.1355
O-3	0	0	.7669	.1210	0	0	.1121
O-4	0	0	0	.8605	.1005	0	.0390
O-5	0	0	0	0	.8595	.0615	.0790
O-6	0	0	0	0	0	.8378	.1622

way of changing promotion rates if we simply want to get to and maintain the desired levels. It should be stressed, that once a specified force level has been shown to be maintainable, it can be attained from any starting force level as was stated in Ref. 1. In the Supply Corps, present rank levels are not equal to desired rank levels. However, if promotion rates were modified to those suggested by P_2, we would eventually reach a system of manpower stocks whose expected value in each rank would equal the desired levels.

A technique that suggests a method to speed up the process of convergence from present levels to desired levels will be discussed in a later chapter.

III. THE DIFFUSION MODEL

The system we wish to model is as depicted below:



The variables of interest are as follows:

Q_o $\equiv$ the force available for recruitment, assumed to be large.

$a(t)$ $\equiv$ the accession rate at time t

$u_i(t)$ $\equiv$ the promotion rate for all members in rank i at time t

$v_i(t)$ $\equiv$ the attrition rate for all members in rank i at time t

$Q_i(t)$ $\equiv$ the Random Variable which is the force level in rank i at time t.

Other variables of interest to be incorporated into the model are:

$q_i(t)$ $\equiv$ the expected fraction of the total force in rank i at time t (computed as fraction of original total force level)

$$\equiv \frac{E[Q_i(t)]}{Q_o}$$

$W_i^P(t)$ $\equiv$ Promotion random noise variable, distributed $N(0,1)$ (when $i = 0$, $W_o^P(t)$ relates to accession)

$W_i^L(t)$ $\equiv$ Loss random noise variable, distributed $N(0,1)$

$X_i(t)$ $\equiv$ standardized random noise variable.

The probability that a member of rank i gets promoted at time t is equal to $u_i(t)$; this implies that the probability that the member does not get promoted is equal to $1 - u_i(t)$. We will assume that whether or not an individual is promoted is independent of others in the system at time t. Thus, the number promoted will have a binomial distribution with expected value $u_i(t)Q_i(t)$ and a variance of $u_i(t)[1 - u_i(t)]Q_i(t)$.

Using similar reasoning the number of attritions in rank i at time t is assumed to have a binomial distribution with an expected value of $v_i(t)Q_i(t)$ and a variance of $v_i(t)[1 - v_i(t)]Q_i(t)$.

In rank 1 we have a special case, since we must consider the accession rate as a function of the total force level. We will assume that there is a large population of potential recruits of size Q_o. We assume the number of people that enter the system at time t has a binomial distribution with mean $a(t)Q_o$ and a variance of $a(t)[1 - a(t)]Q_o$. In what follows we will usually take Q_o to be the initial force strength.

If one deals with a system which is sufficiently large, one can invoke the Central Limit Theorem. Therefore if Q_o is "large", this would imply that $Q_i(t)$ is "large" and therefore the distribution of number promoted can be approximated by a Normal Distribution with mean $u_i(t)Q_i(t)$ and a variance of $u_i(t)[1 - u_i(t)]Q_i(t)$. Similar results apply to the number of attritions in each rank.

Now, recall if X is distributed normal with mean μ and variance $\sigma^2 [X \sim N(\mu,\sigma^2)]$. Then

$$aX + b \sim N(a\mu + b, a^2\sigma^2).$$

Let

$$W_i^P(t) \sim N(0,1) ,$$

then

$$Q_i(t)u_i(t) + \sqrt{Q_i(t)u_i(t)[1 - u_i(t)]}\, W_i^P(t)$$

$$\sim N(Q_i(t)u_i(t), Q_i(t)u_i(t)[1 - u_i(t)])$$

which is a normal distribution with mean and variance equal to the mean and variance specified for the binomial distribution of the number promoted at time t.

Similarly,

$$Q_i(t)v_i(t) + \sqrt{Q_i(t)v_i(t)[1 - v_i(t)]}\; W_i^L(t)$$

$$\sim \quad N(Q_i(t)v_i(t), Q_i(t)v_i(t)[1 - v_i(t)])$$

is a normally distributed random variable with mean and variance equal to the mean and variance specified for the binomial distribution of the number attrited at time t.

For each interval of time (t,t+1] we can define the change in rank i, i.e., $Q_i(t+1) - Q_i(t)$, as a combination of normally distributed random variables.

$$Q_i(t+1) - Q_i(t) \quad =$$

$$u_{i-1}Q_{i-1}(t) + \sqrt{u_{i-1}(t)[1 - u_{i-1}(t)]Q_{i-1}(t)}\; W_{i-1}^P(t)$$

(promotions into rank Q_i)

$$- \; u_i(t)Q_i(t) - \sqrt{u_i(t)[1 - u_i(t)]Q_i(t)}\; W_i^P(t)$$

(promotions out of rank Q_i)

$$- \; v_i(t)Q_i(t) - \sqrt{v_i(t)[1 - v_i(t)]Q_i(t)}\; W_i^L(t)$$

(attritions out of rank Q_i)

From this relationship, $Q_i(t+1)$ can be solved for in terms of $Q_i(t)$. Hence, we assume for $i > 1$,

$$Q_i(t+1) = u_{i-1}(t)Q_{i-1}(t) + \sqrt{u_{i-1}(t)[1 - u_{i-1}(t)]Q_{i-1}(t)}\; W_{i-1}^{P}(t)$$

$$- \sqrt{u_i(t)[1 - u_i(t)]Q_i(t)}\; W_i^{P}(t)$$

$$- \sqrt{v_i(t)[1 - v_i(t)]Q_i(t)}\; W_i^{L}(t)$$

$$+ [1 - u_i(t) - v_i(t)]\; Q_i(t) \qquad \text{(III.1)}$$

where $\{W_i^P(t),\ i = 0,1,\ldots,r-1\}$ and $\{W_i^L(t),\ i = 1,\ldots,r\}$ are independent random variables, having a normal distribution with mean zero and variance equal to one. In addition all the W random variables are independent of $\{Q_i(t),\ i = 1,\ldots,r\}$. Let

$$X_i(t) = \frac{Q_i(t) - Q_o q_i(t)}{\sqrt{Q_o}} \qquad \text{(III.2)}$$

If Q_o is sufficiently large, then $\underset{\sim}{X}(t) = [X_1(t),X_2(t),\ldots,X_r(t)]$ should have approximately a multivariate normal distribution with mean zero.

From equation (III.2) we have

$$Q_i(t) = Q_o q_i(t) + \sqrt{Q_o}\; X_i(t) \qquad i = 1,\ldots,r \qquad \text{(III.3)}$$

In Appendix B we replace the variables $Q_i(t)$ in equation (III.1) by the expression in (III.3) and then take the limit as $Q_o \to \infty$. This procedure yields recursive equations for

$\{q_i(t), i = 1,\ldots,r\}$ and $\{X_i(t), i = 1,\ldots,r\}$. The $\{q_i(t), i = 1,\ldots,r\}$ are constants and $\{X_i(t), i = 1,\ldots,r\}$ is a multivariate normal random variable with mean zero. Hence in light of (III.3), $\underset{\sim}{Q}(t)$ will also have a multivariate normal distribution, at least in the limit as $Q_o \to \infty$. In the next chapter we will present a recursive scheme to evaluate $q_i(t)$, $\mathrm{Var}[X_i(t)]$ and $\mathrm{cov}[X_i(t), X_j(t)]$.

IV. THE RECURSIVE SCHEME

Only results will be presented in this section. Derivations of the results will be presented in Appendix B of this paper.

The following are definitions required for the recursions which follow:

$$\mathrm{Var}[X_i(t)] \equiv m_i(t)$$

$$\mathrm{Cov}[X_i(t), X_j(t)] \equiv m_{i,j}(t)$$

RESULT 1: A RECURSIVE SCHEME FOR COMPUTING $q_i(t)$, $t = 1,2,\ldots$

$i = 1$: $$q_1(t+1) = a(t) + [1 - u_1(t) - v_1(t)]\, q_1(t)$$

$i > 1$: $$q_i(t+1) = u_{i-1}(t) q_{i-1}(t) + [1 - u_i(t) - v_i(t)]\, q_i(t)$$

RESULT 2: A RECURSIVE SCHEME FOR COMPUTING $\mathrm{Var}[X_i(t)]$, $t = 1,2,\ldots$

$i = 1$: $$m_1(t+1) = [1 - u_1(t) - v_1(t)]^2\, m_1(t)$$
$$+\ a(t)[1 - a(t)]$$
$$+\ \{u_1(t)[1 - u_1(t)] + v_1(t)[1 - v_1(t)]\} q_1(t)$$

$$i > 1: \quad m_1(t+1) = [u_{i-1}(t)]^2 \, m_{i-1}(t)$$

$$+ [1 - u_i(t) - v_i(t)]^2 \, m_i(t) + u_{i-1}[1 - u_{i-1}(t)]q_{i-1}(t)$$

$$+ \{u_i(t)[1 - u_i(t)] + v_i(t)[1 - v_i(t)]\}q_i(t)$$

$$+ 2u_{i-1}(t)[1 - u_i(t) - v_i(t)]m_{i-1,i}(t)$$

RESULT 3: A RECURSIVE SCHEME FOR COMPUTING $\mathrm{COV}[X_i(t), X_j(t)]$, $t = 1, 2, \ldots$

<u>$i \neq 1$ and $j = i+1$:</u>

$$m_{i,j}(t+1) = u_{i-1}(t)u_i(t)m_{i-1,i}(t)$$

$$+ u_{i-1}(t)[1 - u_j(t) - v_j(t)]m_{i-1,j}(t) + u_i(t)$$

$$\cdot [1 - u_i(t) - v_i(t)]m_i(t)$$

$$+ [1 - u_i(t) - v_i(t)][1 - u_j(t) - v_j(t)]m_{i,j}(t)$$

$$- u_i(t)[1 - u_i(t)]q_i(t)$$

<u>$i,j \neq 1$ and $j \neq i+1$</u>

$$m_{i,j}(t+1) = u_{i-1}(t)u_{j-1}(t)m_{i-1,j-1}(t)$$

$$+ u_{i-1}(t)[1 - u_j(t) - v_j(t)]m_{i-1,j}(t)$$

$$+ u_{j-1}(t)[1 - u_i(t) - v_i(t)]m_{i,j-1}(t)$$

$$+ [1 - u_i(t) - v_i(t)][1 - u_j(t) - v_j(t)]m_{ij}(t)$$

$\underline{i = 1 \text{ and } j \neq 2}$

$$m_{1,j}(t) = u_{j-1}(t)[1 - u_1(t) - v_1(t)]m_{1,j-1}(t)$$

$$+ [1 - u_1(t) - v_1(t)]m_{1,j}(t)$$

It should be noted that since the

$$\mathrm{Cov}[X_1(t),X_2(t)] = E[X_1(t)X_2(t)] - E[X_1(t)]E[X_2(t)]$$

and since $X_i(t)$ is distributed $N(0,\sigma^2)$; the $\mathrm{Cov}[X_1(t),X_2(t)]$ is equal to the mixed second moment, designated $m_{12}(t)$. Similarly, the variance of $X_i(t)$ is equal to the second moment of $X_i(t) = E[X_i^2(t)] = m_i(t)$.

Given $q_i(t)$ and $m_i(t)$ we have the distribution of $Q_i(t)$, since each $Q_i(t)$ is distributed Normally with a mean equal to $Q_o q_i(t)$ and a variance equal to $Q_o m_i(t)$. As a result of the possible correlation between ranks, of greater importance to us is the Multivariate Normal Distribution of $\underset{\sim}{Q}(t)$. No routine was available that provided the capability of computing the probability of maintaining all the ranks within certain bounds; however, a bivariate normal distribution was used to compute probabilities of maintaining adjacent ranks within plus and minus one standard deviation from the expected rank level. The reason for desiring this capability will be discussed in Chapter VI.

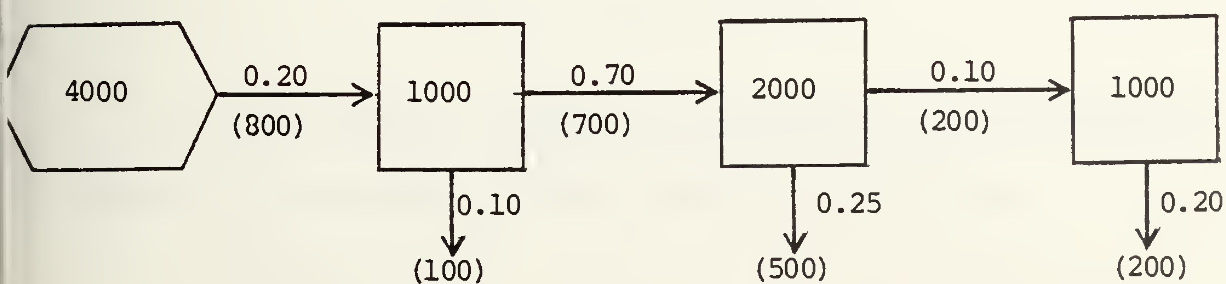
V. BINOMIAL SIMULATION VS. THE DIFFUSION APPROXIMATION

Prior to relying on any approximation of a real world system, a user should have a feel for how close the approximation comes to reflecting the true system. In this chapter we will simulate the year to year operation of an imaginary Markovian 3-rank system. The system will be operating with promotion, attrition and accession rates such that the expected value of each rank is constant, i.e., the system is maintainable at some specified level. The starting parameters are listed below, both in tabular and graphical form.

	$Q_i(t)$ # of individuals	$u_i(t)$ Promotion Rate	$v_i(t)$ Attrition Rate
Rank 1	1000	0.70	0.10
Rank 2	2000	0.10	0.25
Rank 3	1000	0.00	0.20

The accession rate is specified as 0.20 of an available body of recruits estimated at 4000 individuals each year.

Graphically the system would appear as depicted below:



The numbers in the parentheses are the expected number of individuals flowing across the appropriate arcs during each time period. Once again we note, that in a deterministic sense, the system is in exact balance, i.e., the number leaving each rank is exactly equal to the number flowing into the rank.

Let

$Q_o \equiv$ total number in the system

$Q_i(t) \equiv$ number of individuals in rank i at time t

$P_i(t) \equiv$ number of individuals promoted from rank i at time t

$W_i(t) \equiv$ number of individuals attrited from rank i at time t

$u_i(t) \equiv$ promotion rate for rank i at time t

$v_i(t) \equiv$ attrition rate for rank i at time t

$a(t) \equiv$ accession rate at time t.

Note that $P_i(t)$ and $W_i(t)$ are not independent random variables since

$$P_i(t) + W_i(t) \leq Q_i(t)$$

A program BIMOD, outlined in Appendix C, was designed to simulate this system. BIMOD operates as follows:

1. The program is designed to replicate the operation of a manpower system, for a user specified number of times. (Maximum number of replications allowed is 1000.) In the results which will follow, the number of independent replications used was 300. A replication consists of 35 years of promotions, accessions and attritions of the manpower system under consideration. The time increment in each block, corresponds to "t" in all the variables mentioned above. The reason blocks of 35 years were chosen for comparison with the approximation, results from the fact that the diffusion approximation reaches a constant covariance matrix at the 35 year point. By terminating each replication at 35 years, and storing the value of each rank at the end of each run, we can then compute an estimate for the covariance matrix of the simulation at the 35 year point.

2. For each time period t, BIMOD generates a binomial random deviate with parameters $Q_i(t)$ and $[u_i(t) + v_i(t)]$. This deviate represents the total number of individuals leaving rank i at time t; both through promotion and through attrition. Call this random variable $L_i(t)$, i = 1, ..., r.

3. For each time period, we also generate independent, binomial random deviates with parameters $L_i(t)$ and $u_i(t)/[U_i(t) + v_i(t)]$, i = 1,...,r; and Q_o and a(t). These deviates represent the total number of promotions from rank i at time t, $P_i(t)$; and the total number of accessions into the system at time t, A(t).

4. The $Q_i(t)$ are updated for each time period utilizing the following recursive equations

$$Q_1(t+1) = Q_1(t) - L_1(t) + A(t)$$

$$Q_i(t+1) = Q_i(t) - L_i(t) + P_{i-1}(t) \quad \text{for} \quad i = 2,3$$

In comparing the simulation against the diffusion approximation, we would like to reduce the comparison to the random terms. We do this in order to determine how good the approximation is in predicting the means, the variances, and the covariances of the random terms.

As was discussed in Chapter III, equation (III.3) is the cornerstone of the diffusion approximation

$$Q_i(t) = Q_o q_i(t) + \sqrt{Q_o}\, X_i(t). \qquad \text{(IV.1)}$$

We could describe the system of our simulation in similar terms,

$$M_i(t) = Q_o m_i(t) + \sqrt{Q_o}\, R_i(t) \qquad \text{(IV.2)}$$

where $Q_o m_i(t)$ is the expected value of rank i at time t, $M_i(t)$ is the observed value of rank i at time t, and $R_i(t)$ is the random residual term for rank i at time t. The comparison of (IV.1) and (IV.2) would commence with an examination of the $R_i(t)$ residual terms

$$R_i(t) = \frac{M_i(t) - Q_o m_i(t)}{\sqrt{Q_o}} \qquad \text{(IV.3)}$$

To see if the $R_i(t)$ do in fact exhibit an approximate normal distribution with mean equal to zero.

BIMOD was run with the following input values:

NUMBER OF REPLICATIONS		300
YEARS CONSIDERED IN EACH REPLICATION		35
INITIAL FORCE LEVELS		
	Q_1	1000
	Q_2	2000
	Q_3	1000
PROMOTION RATES		
	$u_1(t)$	0.70
	$u_2(t)$	0.10
	$u_3(t)$	0.00
ACCESSION RATES		
	$a(t)$	0.20
ATTRITION RATES		
	$v_1(t)$	0.10
	$v_2(t)$	0.25
	$v_3(t)$	0.20

At the end of each 35 year replication, the value of each rank was stored, and the residual values were computed as in equation (IV.3). The expected values used in equation (IV.3) were the initial force levels as stipulated above, since they are the levels we expect to maintain. When the 300th replication was completed, the mean value of the residuals in each rank, $\overline{R}_i$, was computed, and this in turn was used to

compute the estimate for the covariance matrix of the 35th year residuals. The formula used in computing the covariance matrix is

$$\mathrm{Cov}(R_i, R_j) = \sum \frac{(R_i - \bar{R}_i)(R_j - \bar{R}_j)}{n-2}$$

where the R_i are the stored values at the end of each replication.

The following is a compilation of the results obtained from the simulation.

	$E[R_i(t)]$	Kurtosis	Skewness
Rank 1	0.0398	2.7468	0.2054
Rank 2	0.0162	2.6670	0.0248
Rank 3	0.0041	2.8470	0.0708

Kurtosis and skewness information was obtained from a histogram of the total run residuals. It should be noted that the Kurtosis Coefficient for a normally distributed random variable is equal to 3.0 and the skewness should equal 0.

This first level of comparison of the $R_i(t)$ terms with the $X_i(t)$ terms seems to indicate that the assumptions of normality with mean equal to zero does in fact provide a reasonable approximation, with a possible exception of rank 1, as indicated by its skewness coefficient.

The next level of comparison between $R_i(t)$ and $X_i(t)$ involves the predicted variances and covariances. Utilizing BIMOD the following covariance matrix was obtained for the $R_i(t)$ residual terms.

	Rank 1	Rank 2	Rank 3
Rank 1	0.2035	0.0000	-0.0022
Rank 2	0.0000	0.4747	0.0245
Rank 3	-0.0022	0.0245	0.2783

and this is compared to the following $X_i(t)$ covariance matrix obtained from the diffusion approximation program MNPWR, outlined in Appendix C.

	Rank 1	Rank 2	Rank 3
Rank 1	0.2448	-0.0210	-0.0005
Rank 2	-0.0210	0.5059	-0.0289
Rank 3	-0.0005	-0.0289	0.2373

At first glance, it would appear that the approximations for the covariance terms did fairly well. Of particular note, however, is the fact that the approximation specifies a negative correlation between ranks 2 and 3, while the simulation specifies a positive correlation for the same ranks.

In order to better understand what is going on, the variance of the covariance terms, i.e., the variance of the $\frac{(R_i - \bar{R}_i)(R_j - \bar{R}_j)}{n-2}$ for all end of run R_i, was computed to

determine whether the diffusion approximation estimate falls in the range $Cov(R_i,R_j) \pm Std\ Dev[Cov(R_i,R_j)]$. The following are the variance results.

Ranks	Cov - 1 Std Dev	Simulation Covariance	Cov + 1 Std Dev
1,2	-0.0173	0.0000	0.0173
1,3	-0.0162	-0.0022	0.0118
2,3	0.0047	0.0245	0.0443

While the covariance range specified above still implies a positive covariance for ranks 2 and 3, the covariance values under consideration are small enough to imply that the approximation estimate for the covariance is quite good.

The next important matter of note, is the fact that in both matrices above, the covariance terms are essentially zero; and the fact that ranks appear to be independent of one another is borne out by very low correlation coefficients between ranks, both in the simulated system and in the diffusion approximation.

A possible explanation for the resulting independence could be that since we do not vary promotion, attrition, or accession rates in either the simulation or the diffusion approximation, the probability laws of the models are the same as those in a model in which the motions of individuals through the system are independent. It is known that if the input random variables are independent and have Poisson distributions and the motions of the individual in the system

are independent, then the number of individuals in stage i at time t, $Y_i(t)$, are asymptotically independent with marginal Poisson distributions as $t \to \infty$ [Refs. 1,8].

In comparing the variances, we note that the diffusion approximation provides reasonable estimates for the variance of the simulation residuals.

In conclusion, one could possibly simulate a 6-rank system and obtain better results for the variances and covariances of the ranks. However, due to the enormous number of deviate generations required to produce representative output, it is felt that the much simpler computations of the diffusion approximation far outweigh their loss of accuracy.

VI. DIFFUSION APPROXIMATION RESULTS

In this final chapter, we will present the results obtained when the 6-rank Supply Corps system is modeled utilizing the Diffusion Approximation Program, listed in Appendix C. In addition a program called MINI will be discussed and its results reintroduced into a second run of the Diffusion Approximation to the Supply Corps system. The final subsection of this chapter will provide some recommendations for further study.

A. A DIFFUSION APPROXIMATION APPLIED TO THE PRESENT LEVELS OF THE SUPPLY CORPS

In Chapter II, we determined, at least in the deterministic sense, that under specific assumptions, desired rank levels, as defined by Ref. 9, could be maintained. We now wish to utilize those same attrition rates

$$\underset{\sim}{V}(t) = [0.0272, 0.1355, 0.1121, 0.0390, 0.0790, 0.1622]$$

in the Diffusion Approximation and to apply them to present Supply Corps force levels. These levels are presented below:

Rank	Level
O-1	632
O-2	691
O-3	961
O-4	792
O-5	616
O-6	177

These values were obtained from the Supply Corps Officer Personnel section of the Naval Supply Systems Command, and were current values for 310X officers as of September 1979.

The program MNPWR listed in Appendix C, incorporates the recursive equations derived in Chapter IV. A feature of this program is that, given the present levels, desired levels, and estimates for attrition rates specified previously, the program will compute the following fixed promotion rates,

$$\underset{\sim}{U} = [0.5099, 0.3711, 0.1210, 0.1005, 0.0615, 0.0000]$$

and a fixed accession rate "a" of 0.0862. Note that these promotion and accession rates are those which are necessary to attain and maintain the desired force levels given that we start at the present force levels, and are the same as those specified in Section II.B of this thesis.

Therefore, in the first of a series of computations, we attempt to see how long it will take to attain the desired force levels using the promotion and accession rates specified above. Note, we are not worried about maintainability, since Ref. 1 states for the deterministic case, that if a structure is maintainable, it is also attainable. However, in the stochastic sense maintainability has a slightly different connotation. In considering methods of control, we are now interested in methods that not only provide the expected rank levels, but also provide manpower planners estimates for

the variance of each rank. Returning to the initial computation, the reason we are not concerned about maintainability is that if the program is able to compute non-negative accession and promotion rates, then the structure is maintainable.

A program run was made to determine how long it would take to attain the desired force structure. At a point 50 years into the future, the expected number in each rank was approximately at the desired level. This length of time is, without much thought, unacceptable. A technique which allows faster convergence to the desired levels will be described in the next subsection.

The following is a listing of the expected values and the covariance matrix of the system at time 50.

EXPECTED VALUE OF EACH RANK AT TIME 50

Rank	E[Rank]	Desired
O-1	620.99927	621
O-2	624.99902	625
O-3	994.99780	995
O-4	862.98999	863
O-5	616.91138	617
O-6	233.85883	234

THE COVARIANCE MATRIX FOR FORCE LEVELS AT TIME EQUAL TO 50

	O-1	O-2	O-3	O-4	O-5	O-6
O-1	606.33521	-15.64740	- 4.16817	- 0.38805	- 0.02998	- 0.00140
O-2	-15.64740	692.62646	-38.01680	- 4.68668	- 0.46078	- 0.02646
O-3	- 4.16817	-38.01680	1030.59033	-39.38757	- 9.84769	- 1.35218
O-4	- 0.38805	- 4.68668	-39.38757	859.45459	-19.93602	- 4.54186
O-5	- 0.02998	- 0.46078	- 9.84769	-19.93602	626.51392	-10.72863
O-6	- 0.00140	- 0.02646	- 1.35218	- 4.54186	-10.72863	230.24536

Analysis of the covariance matrix will be reserved until the next subsection.

B. MINIMUM CONVERGENCE TIME

The quickest method of attaining the desired force from the present force would be simply to promote (or attrit) all shortages (or excesses) in one time period. In the case being examined, this would be totally infeasible both for legislative and for practical purposes. To attain the desired levels in one time period would involve a very large number of officer promotions, which would be totally unacceptable from the organizational standpoint.

The approach taken in program MINI (see Appendix C), is a Dynamic Programming Approach. Given what are reasonable bounds for promotion and accession rates, the program will minimize the difference from present levels to desired levels, at each time period, for specified attrition rates.

Utilizing the following bounds for accession and promotion rates,

	Lower Bounds (accession and promotion rates required to maintain the system)	Upper Bounds (110% of the lower bound rates)
accession	0.0862	0.0948
O-1	0.5099	0.5609
O-2	0.3711	0.4082
O-3	0.1210	0.1331
O-4	0.1005	0.1106
O-5	0.0615	0.0677
O-6	0.0000	0.0000

and the attrition rates specified at the beginning of this chapter, we could reduce the convergence time required for the expected number of individuals in each rank to attain the desired level from about 50 years to at most 8 years.

CONVERGENCE TIME REQUIRED

Rank	Time Required for Convergence
O-1	4 years
O-2	3 years
O-3	1 year
O-4	8 years
O-5	6 years
O-6	8 years

As an output of program MINI, we obtain the promotion rates required to minimize the shortfall at each time period. The following is a listing of the promotion rates suggested by MINI.

MINI PROMOTION
AND ACCESSION RATES

TIME	O-1	O-2	O-3	O-4	O-5	ACCESSION RATE
1	0.5099	0.3902	0.1331	0.1106	0.0677	0.0862
2	0.5099	0.3754	0.1331	0.1106	0.0677	0.0862
3	0.5104	0.3829	0.1331	0.1106	0.0677	0.0862
4	0.5286	0.3904	0.1331	0.1106	0.0677	0.0891
5	0.5292	0.3904	0.1331	0.1106	0.0677	0.0893
6	0.5292	0.3904	0.1331	0.1088	0.0677	0.0893
7	0.5292	0.3904	0.1331	0.1066	0.0677	0.0893
8	0.5179	0.3790	0.1260	0.1015	0.0622	0.0875

Note that between years 7 and 8 the promotion and accession rates begin to decrease. This is a direct result of the fact that we are approaching the desired levels, and following year 8, we then use $\underset{\sim}{u}$ and "a" as specified in Section VI.A.

Utilizing these computed accession and promotion rates as input parameters to the Diffusion Approximation yields information on how the system would react to system controls. The following is a summary of the Diffusion Approximation results at the 8 year point.

Rank	Desired Level	Expected Value Approximation Result
O-1	621	621.00903
O-2	625	624.95386
O-3	995	994.87744
O-4	863	863.02441
O-5	617	617.00854
O-6	234	233.97964

Of even greater significance are the variance/covariance results. Below is the covariance matrix obtained at the 8 year point. Variances for each rank are on the main diagonal.

A striking result becomes evident if one compares the expected value and the variance of each rank. The independent motions property is still evident, even with promotion control, i.e., the expected value in each rank is approximately equal to the variance of the rank (as one would expect for Poisson distributed random variables). What this may imply,

FORCE LEVEL COVARIANCE MATRIX

	O-1	O-2	O-3	O-4	O-5	O-6
O-1	605.5386	-16.1089	- 4.2081	- 0.3800	- 0.0258	- 0.0009
O-2	-16.1089	629.9961	-38.5305	- 4.7504	- 0.4241	- 0.0203
O-3	- 4.2081	-38.5305	1035.1289	-37.5407	- 8.0922	- 0.8127
O-4	- 0.3800	- 4.7504	-37.5407	864.9128	-13.7738	- 2.1370
O-5	- 0.0258	- 0.4241	- 8.0922	-13.7738	633.1138	- 8.3675
O-6	- 0.0009	- 0.0203	- 0.8127	- 2.1270	- 8.3675	231.6361

is that in this case even though we vary the promotion and accession rates, the changes allowed in those rates are much too small to counteract the independent motions property. As a result, bivariate probabilities taken indicated that the ranks are essentially independent.

If in fact the ranks are relatively independent, one could then compute expected value plus and minus one standard deviation of force level in each rank, and know that with the assumption of a normal distribution for force levels, we have approximately a 68% chance of being in that interval. From our calculations we get the following results for upper and lower bounds of force level.

Rank	Lower Bound	Desired Level	Upper Bound
O-1	596	621	645
O-2	598	625	651
O-3	962	995	1027
O-4	833	863	892
O-5	591	617	642
O-6	218	234	249

Given these values, manpower planners need not overreact to random deviations in force levels within the above bounds. If, for example, the O-4 rank were to fall to as low as 833, this would not necessarily imply that we should increase next year's accessions; it could be rationalized as a reasonable value as compared to the variance computed by the Diffusion Approximation. Without the variance estimates, planners

do not possess the tools necessary to determine whether or not fluctuations in each rank can be explained by the internal randomness of the System, or whether the fluctuations are due to external forces which require changes in accession and/or promotion rates.

C. RECOMMENDATIONS FOR FURTHER STUDY

The Diffusion Approximation hinges on reasonable estimates for the attrition rate of each rank. Therefore, Recommendation 1 would be to derive a useful estimating relationship for the forecasting of attrition rates.

Another possibility for an extension of this thesis, would be to consider confidence intervals for attrition rates, and to input these intervals with the Diffusion Approximation. Analysis of the results obtained would be interesting in regards to the independent motions observed when point estimates were used for attrition rate.

It is clear from the foregoing analysis that maintaining a grade structure in a stochastic environment is much less straightforward than a deterministic analysis of the problem would suggest. Under rather special conditions we have seen that it is possible to maintain a structure in spite of random variation in each rank. With better forecasting of attrition rates, it is felt that the Diffusion Approximation will prove to be an invaluable tool to manpower planners.

APPENDIX A

DESIGNATOR 3100
AUTHORIZATIONS SUPPLY CORPS (SC)

DESIGNATOR 3100 AUTHORIZATIONS
SUPPLY CORPS (SC)

GRADE	FY 1979	FY 1980	FY 1981	FY 1982	FY 1983	FAC* T	FAC* E	FAC* R
FLAG	26	26	26	26	26			
O-6 CAPT	237	237	235	230	230	3		
O-5 CDR	620	620	614	615	615	7	4	
O-4 LCDR	862	862	858	863	870	15	13	2
O-3 LT	968	980	990	1009	1029	2	10	
O-2 LTJG	625	630	622	622	625		15	
O-1 ENS	667	620	605	610	605		6	1
TOTAL	4005	3975	3950	3975	4000	27	48	3

* INCLUDED WITHIN TOTAL AUTHORIZATIONS

1-33 .

APPENDIX B

DERIVATIONS

B.1 DERIVATION OF THE $q_i(t+1)$ RECURSION, THE VARIANCE RECURSION, AND THE RELATIONSHIPS FOR $X_i(t)$

$$Q_1(t+1) - Q_1(t) = a(t)Q_o + \sqrt{a(t)[1 - a(t)]Q_o}\, W_o{}^P(t)$$

$$- u_1(t)Q_1(t) - \sqrt{u_1(t)[1 - u_1(t)]Q_1(t)}\, W_1{}^P(t)$$

$$- v_1(t)Q_1(t) - \sqrt{v_1(t)[1 - v_1(t)]Q_1(t)}\, W_1{}^L(t)$$

(for definition of the mentioned variables, see Chapter IV). Substituting $\sqrt{Q_o}\, X_1(t) + Q_o q_1(t)$ for $Q_1(t)$ and $\sqrt{Q_o}\, X_1(t+1) + Q_o q_1(t+1)$ for $Q_1(t+1)$, we have

$$\sqrt{Q_o}[X_1(t+1) - X_1(t) + \sqrt{Q_o}\, q_1(t+1) - \sqrt{Q_o}\, q_1(t)] =$$

$$\sqrt{Q_o}\{\sqrt{Q_o}\, a(t) + \sqrt{a(t)[1 - a(t)}\, W_o{}^P(t)$$

$$- u_1(t)[X_1(t) + \sqrt{Q_o}\, q_1(t)]$$

$$- \sqrt{u_1(t)[1 - u_1(t)][(1/\sqrt{Q_o})X_1(t) + q_1(t)]}\, W_1{}^P(t)$$

$$- v_1(t)[X_1(t) + \sqrt{Q_o}\, q_1(t)]$$

$$- \sqrt{v_1(t)[1 - v_1(t)][(1/\sqrt{Q_o})X_1(t) + q_1(t)]}\, W_1{}^L(t)\}$$

$$X_1(t+1) - X_1(t) =$$

$$\sqrt{Q_o}\, q_1(t) - \sqrt{Q_o}\, q_1(t+1) + \sqrt{Q_o}\, a(t) + \sqrt{a(t)[1-a(t)]}\; W_o^P(t)$$

$$- u_1(t)\, X_1(t) - u_1(t)\, \sqrt{Q_o}\, q_1(t)$$

$$- \sqrt{u_1(t)[1-u_1(t)][(1/\sqrt{Q_o})X_1(t) + q_1(t)]}\; W_1^P(t)$$

$$- v_1(t)\, X_1(t) + v_1(t)\, \sqrt{Q_o}\, q_1(t)$$

$$- \sqrt{v_1(t)[1-v_1(t)][(1/\sqrt{Q_o})X_1(t) + q_1(t)]}\; W_1^L(t)$$

Heuristically, since $X_1(t)$ is a normal random variable, $X_1(t+1) - X_1(t)$ must be finite and therefore any terms with $\sqrt{Q_o}$ must approach zero as $Q_o \to \infty$. Therefore in the limit as $Q_o \to \infty$

$$X_1(t+1) - X_1(t) = \sqrt{a(t)[1-a(t)]}\; W_o^P(t)$$

$$- u_1(t)\, X_1(t) - \sqrt{u_1(t)[1-u_1(t)]q_1(t)}\; W_1^P(t)$$

$$- v_1(t)\, X_1(t) - \sqrt{v_1(t)[1-v_1(t)]q_1(t)}\; W_1^L(t)$$

Rewriting, we have

$$X_1(t+1) = [1 - u_1(t) - v_1(t)]\, X_1(t)$$

$$+ \sqrt{a(t)[1 - a(t)]}\; W_o^P(t)$$

$$- \sqrt{u_1(t)[1 - u_1(t)]q_1(t)}\; W_1^P(t)$$

$$- \sqrt{v_1(t)[1 - v_1(t)]q_1(t)}\; W_1^L(t) \qquad \text{(B.1.1)}$$

It also follows that in the limit as $Q_o \to \infty$

$$q_1(t+1) - q_1(t) = a(t) - \{u_1(t) + v_1(t)\}q_1(t)$$

or

$$q_1(t+1) = a(t) + [1 - u_1(t) - v_1(t)]q_1(t) \qquad \text{(B.1.2)}$$

Using a similar approach for $Q_i(t+1) - Q_i(t)$, $i \neq 1$, the following result was obtained in the limit as $Q_o \to \infty$.

$$X_i(t+1) = u_{i-1}(t)X_{i-1}(t) + [1 - u_i(t) - v_i(t)]\, X_i(t)$$

$$+ \sqrt{u_{i-1}(t)[1 - u_{i-1}(t)]q_{i-1}(t)}\; W_{i-1}^P(t)$$

$$- \sqrt{u_i(t)[1 - u_i(t)]q_i(t)}\; W_i^P(t)$$

$$- \sqrt{v_i(t)[1 - v_i(t)]q_i(t)}\; W_i^L(t) \qquad \text{(B.1.3)}$$

and

$$q_i(t+1) - q_i(t) = u_{i-1}(t)q_{i-1}(t) - [u_i(t) + v_i(t)]q_i(t)$$

or

$$q_i(t+1) = u_{i-1}(t)q_{i-1}(t) + [1 - u_i(t) - v_i(t)]q_i(t) \tag{B.1.4}$$

We now have expressions for all the $X_i(t+1)$ and $q_i(t+1)$ in terms of $X_i(t)$ and $q_i(t)$ respectively.

In order to calculate the variances and covariances of the $\{X_i(t), i = 1,\dots,r\}$, we must first determine the first and second moments of the $X_i(t+1)$.

$$\begin{aligned} E[X_1(t+1)] = {} & [1 - u_1(t) - v_1(t)]\, E[X_1(t)] \\ & + \sqrt{a(t)[1 - a(t)]}\, E[W_o^P(t)] \\ & - \sqrt{u_1(t)[1 - u_1(t)]q_1(t)}\, E[W_1^P(t)] \\ & - \sqrt{v_1(t)[1 - v_1(t)]q_1(t)}\, E[W_1^L(t)] \end{aligned} \tag{B.1.5}$$

Recall that $\{W_i^P(t), i = 0,1,\dots,r-1\}$ and $\{W_i^L(t), i = 1,\dots,r\}$ are normally distributed with a mean of zero. The random variable $X_1(t)$ represents the randomness in the system at time t. Initially, since the system is starting from a known

set of rank levels, $X_1(0)$ is defined to be zero. As a result, equation (B.1.5) implies that $E[X_1(t+1)] = 0$ for all time t. Further,

$$
\begin{aligned}
Var[X_1(t+1)] \;=\; & [1 - u_1(t) - v_1(t)]^2 \; Var[X_1(t)] \\
& + a(t)[1 - a(t)] \; Var[W_o^P(t)] \\
& + u_1(t)[1 - u_1(t)]q_1(t) \; Var[W_1^P(t)] \\
& + v_1(t)[1 - v_1(t)]q_1(t) \; Var[W_1^L(t)]
\end{aligned}
$$

Since the variances of the W_i random variables are equal to one,

$$
\begin{aligned}
Var[X_1(t+1)] \;=\; & [1 - u_1(t) - v_1(t)]^2 \; Var[X_1(t)] \\
& + a(t)[1 - a(t)] + u_1(t)[1 - u_1(t)]q_1(t) \\
& + v_1(t)[1 - v_1(t)]q_1(t)
\end{aligned}
$$

Since the expected value of $X_1(t)$ is equal to zero, the variance of $X_1(t)$ is equal to its second moment which we will call $m_1(t)$. Therefore we have the following:

$$
\begin{aligned}
m_1(t+1) \;=\; & [1 - u_1(t) - v_1(t)]^2 \; m_1(t) + a(t)[1 - a(t)] \\
& + \{u_1(t)[1 - u_1(t)] + v_1(t)[1 - v_1(t)]\}q_1(t)
\end{aligned}
$$

Using similar reasoning, we get a general recursive scheme for $i > 1$:

$$
\begin{aligned}
m_i(t+1) = {} & [u_{i-1}(t)]^2\, m_{i-1}(t) \\
& + [1 - u_i(t) - v_i(t)]^2\, m_i(t) \\
& + u_{i-1}(t)[1 - u_{i-1}(t)]\, q_{i-1}(t) \\
& + \{u_i(t)[1 - u_i(t)] + v_i(t)[1 - v_i(t)]\}q_i(t) \\
& + 2\{u_{i-1}(t)[1 - u_i(t) - v_i(t)]\}\, m_{i-1,i}(t)
\end{aligned}
$$

Note, however, for $i > 1$ we get a covariance term. In the next section, we will obtain a recursive scheme for computing the covariances.

B.2 DERIVATION OF COVARIANCE FORMULAS

Recall:

$$
\begin{aligned}
X_1(t+1) = {} & [1 - u_1(t) - v_1(t)]\, X_1(t) \\
& + \sqrt{a(t)[1 - a(t)]}\, W_o{}^P(t) \\
& - \sqrt{u_1(t)[1 - u_1(t)]q_1(t)}\, W_1{}^P(t) \\
& - \sqrt{v_1(t)[1 - v_1(t)]q_1(t)}\, W_1{}^L(t)
\end{aligned}
$$

If we let

$$b_i(t) = 1 - u_i(t) - v_i(t)$$

$$c(t) = a(t)[1 - a(t)]$$

$$h_i(t) = u_i(t)[1 - u_i(t)]$$

$$k_i(t) = v_i(t)[1 - v_i(t)]$$

then

$$X_1(t+1) = b_1(t)\ X_1(t) + \sqrt{c(t)}\ W_o{}^P(t)$$

$$- \sqrt{h_1(t)q_1(t)}\ W_1{}^P(t) - \sqrt{k_1(t)q_1(t)}\ W_1{}^L(t)$$

Similarly recall

$$X_2(t+1) = u_1(t)\ X_1(t) + b_2(t)\ X_2(t)$$

$$+ \sqrt{h_1(t)q_1(t)}\ W_1{}^P(t) - \sqrt{h_2(t)q_2(t)}\ W_2{}^P(t)$$

$$- \sqrt{k_i(t)q_2(t)}\ W_2{}^L(t)$$

It is assumed that at time zero the $\mathrm{Cov}[X_1(t), X_2(t)] = 0$ for $t = 0$,

$$\mathrm{Cov}[X_1(t+1), X_2(t+1)] = E[X_1(t+1)X_2(t+1)] - E[X_1(t+1)]E[X_2(t+1)]$$

$$X_1(t+1)X_2(t+1) =$$

$$[b_1(t)X_1(t) + \sqrt{c(t)}\, W_o{}^P(t) - \sqrt{h_1(t)q_1(t)}\, W_1{}^P(t) - \sqrt{k_1(t)q_1(t)}\, W_1{}^L(t)]$$

$$\cdot [u_1(t)X_1(t) + b_2(t)X_2(t) + \sqrt{h_1(t)q_1(t)}\, W_1{}^P(t) - \sqrt{h_2(t)q_2(t)}\, W_2{}^P(t)$$

$$- \sqrt{k_2(t)q_2(t)}\, W_2{}^L(t)]$$

$$= b_1(t)u_1(t)X_1{}^2(t) + b_1(t)b_2(t)X_1(t)X_2(t) + b_1(t)\sqrt{h_1(t)q_1(t)}\, X_1(t)W_1{}^P(t)$$

$$- b_1(t)\sqrt{h_2(t)q_2(t)}X_1(t)W_2{}^P(t) - b_1(t)\sqrt{k_2(t)q_2(t)}\, X_1(t)W_2{}^L(t)$$

$$+ u_1(t)\sqrt{c(t)}\, X_1(t)W_o{}^P(t) + b_2(t)\sqrt{c(t)}\, X_2(t)W_o{}^P(t)$$

$$+ \sqrt{c(t)h_1(t)q_1(t)}\, W_1{}^P(t)W_o{}^P(t) - \sqrt{c(t)h_2(t)q_2(t)}\, W_2{}^P(t)W_o{}^P(t)$$

$$- \sqrt{c(t)k_2(t)q_2(t)}\, W_2{}^L(t)W_o{}^P(t)$$

$$- u_1(t)\sqrt{h_1(t)q_1(t)}\, X_1(t)W_1{}^P(t) - b_2(t)\sqrt{h_1(t)q_1(t)}\, X_2(t)W_1{}^P(t)$$

$$- h_1(t)q_1(t)\,[W_1{}^P(t)]^2 + \sqrt{h_1(t)q_1(t)h_2(t)q_2(t)}W_2{}^P(t)W_1{}^P(t)$$

$$+ \sqrt{h_1(t)q_1(t)k_2(t)q_2(t)}W_2{}^L(t)W_1{}^P(t) - u_1(t)\sqrt{k_1(t)q_1(t)}X_1(t)W_1{}^L(t)$$

$$- b_2(t)\sqrt{k_1(t)q_1(t)}\, X_2(t)W_1{}^L(t) - q_1(t)\sqrt{h_1(t)k_1(t)}\, W_1{}^P(t)W_1{}^L(t)$$

$$+\sqrt{k_1(t)q_1(t)h_2(t)q_2(t)}W_2{}^P(t)W_1{}^L(t) + \sqrt{k_1(t)q_1(t)k_2(t)q_2(t)}\, W_2{}^L(t)W_1{}^L(t)$$

$$E[X_1(t+1)X_2(t+1)] =$$

$$b_1(t)u_1(t)E[X_1^2(t)] + b_1(t)b_2(t)E[X_1(t)X_2(t)] + b_1(t)\sqrt{h_1(t)q_1(t)}E[X_1(t)W_1^P(t)]$$

$$- b_1(t)\sqrt{h_2(t)q_2(t)}E[X_1(t)W_2^P(t)] - b_1(t)\sqrt{k_2(t)q_2(t)}E[X_1(t)W_2^L(t)]$$

$$+ u_1(t)\sqrt{c(t)}E[X_1(t)W_o^P(t)] + b_2(t)\sqrt{c(t)}E[X_2(t)W_o^P(t)]$$

$$+\sqrt{c(t)h_1(t)q_1(t)}\, E[W_1^P(t)W_o^P(t)] - \sqrt{c(t)h_2(t)q_2(t)}\, E[W_2^P(t)W_o^P(t)]$$

$$-\sqrt{c(t)k_2(t)q_2(t)}\, E[W_2^L(t)W_o^P(t)] - u_1(t)\sqrt{h_1(t)q_1(t)}\, E[X_1(t)W_1^P(t)]$$

$$- b_2(t)\sqrt{h_1(t)q_1(t)}\, E[X_2(t)W_1^P(t)] - h_1(t)q_1(t)E[(W_1^P(t))^2]$$

$$+ \sqrt{h_1(t)q_1(t)h_2(t)q_2(t)}\, E[W_2^P(t)W_1^P(t)]$$

$$- u_1(t)\sqrt{k_1(t)q_1(t)}E[X_1(t)W_1^L(t)] - b_2(t)\sqrt{k_1(t)q_1(t)}E[X_2(t)W_1^L(t)]$$

$$- q_1(t)\sqrt{h_1(t)k_1(t)}\, E[W_1^P(t)W_1^L(t)] + \sqrt{k_1(t)q_1(t)h_2(t)q_2(t)}\, E[W_2^P(t)W_1^L(t)]$$

$$+ \sqrt{k_1(t)q_1(t)k_2(t)q_2(t)}\, E[W_2^L(t)W_1^L(t)]$$

Since the $X_i(t)$ are assumed independent of the $W_i(t)$ and $E[(W_1^P(t))^2] = 1$, the above simplifies to

$$m_{12}(t+1) = E[X_1(t+1)X_2(t+1)] = b_1(t)\ u_1(t)\ m_1(t)$$

$$+ b_1(t)\ b_2(t)\ m_{12}(t) - h_1(t)\ q_1(t)$$

$$m_{12}(t+1) = [1 - u_1(t) - v_1(t)]\ u_1(t)\ m_1(t)$$

$$+ [1 - u_1(t) - v_1(t)][1 - u_2(t) - v_2(t)]\ m_{12}(t)$$

$$- u_1(t)[1 - u_1(t)]\ q_1(t)$$

A similar approach can be taken for all cross product terms, and the following general relationships will be obtained:

<u>$i = 1,\ j > 2$</u>

$$m_{1,j}(t+1) = u_{j-1}(t)[1 - u_1(t) - v_1(t)]\ m_{1,j-1}(t)$$

$$+ [1 - u_1(t) - v_1(t)][1 - u_j(t) - v_j(t)]\ m_{1,j}(t)$$

<u>$i > 1,\ j = i+1$</u>

$$m_{i,j}(t+1) = u_{i-1}(t)\ u_i(t)\ m_{i-1,i}(t)$$

$$+ u_{i-1}(t)[1 - u_j(t) - v_j(t)]\ m_{i-1,j}(t)$$

$$+ u_i(t)[1 - u_i(t) - v_i(t)]\ m_i(t)$$

$$+ [1 - u_i(t) - v_j(t)][1 - u_j(t) - v_j(t)]\ m_{i,j}(t)$$

$$- u_i(t)[1 - u_i(t)]\ q_i(t)$$

$\underline{i,j > 1,\ j \neq i+1}$

$$
\begin{aligned}
m_{i,j}(t+1) = {} & u_{i-1}(t)\, u_{j-1}(t)\, m_{i-1,j-1}(t) \\
& + u_{i-1}(t)\,[1 - u_j(t) - v_j(t)]\, m_{i-1,j}(t) \\
& + u_{j-1}(t)\,[1 - u_i(t) - v_i(t)]\, m_{i,j-1}(t) \\
& + [1 - u_i(t) - v_i(t)]\,[1 - u_j(t) - v_j(t)]\, m_{ij}(t)
\end{aligned}
$$

To fill out the remainder of the covariance terms, recall that

$$m_{ij}(t+1) = m_{ji}(t+1) .$$

```
C                              **************
C                            ***** APPENDIX C *****
C                            * COMPUTER  PROGRAMS *
C                            **********************
C
C
C
C            C.1  BINOMIAL SIMULATION PROGRAM (PROGRAM BIMCD)
C
C
      REAL *8DSEED
      INTEGER RUN,Q,START,SUMX,LAS,P,R,QZERO,RR,XSQ
      DIMENSION U(3), V(3), Q(3), START(3), SUMX(3), LAS(3,1000), P(3),
     1XSQ(3), LV(3), EXP(3), COV(3,3), MEAN(3), RESID(3,1000), SUMXR(3),
     2 SCALE(2), RSQ(3), EXPR(3), R1(1000), R2(1000), R3(1000)
      DSEED = 30290.0D0
      READ (5,1) RUN,A
    1 FORMAT (I4,F10.0)
      WRITE (6,2) RUN,A
    2 FORMAT (//,10X,I10,F10.4)
C
      DO 5 I=1,3
      READ (5,3) Q(I),U(I),V(I)
    3 FORMAT (I4,2F3.0)
      WRITE (6,4) Q(I),U(I),V(I)
    4 FORMAT (//,10X,I10,2F10.4)
      START(I) = Q(I)
      XSQ(I) = 0
      SUMX(I) = 0
      SUMXR(I) = 0.0
      RSQ(I) = 0.0
    5 CONTINUE
C
      MEAN(1) = 1000
      MEAN(2) = 2000
      MEAN(3) = 1000
      QZERO = Q(1)+Q(2)+Q(3)
C
      DO 10 R=1,RUN
      RR = FIX(GGBIR(DSEED,QZERO,A))
C
      DO 7 I=1,3
      IF (I.EQ.3) GO TO 6
      RL = U(I)+V(I)
      LV(I) = FIX(GGBIR(DSEED,Q(I),RL))
      RATE = U(I) / (U(I) + V(I))
      P(I) = FIX(GGBIR(DSEED,Q(I),RATE))
      GO TO 7
```

```
C                 C.1  BINOMIAL SIMULATION PROGRAM (PROGRAM BIMOD)
C
C
C
    6 LV(3) = FIX(GGBIR(DSEED,Q(I),V(I)))
    7 CONTINUE
C
      Q(1) = Q(1)-LV(1)+RR
      Q(2) = Q(2)-LV(2)+P(1)
      Q(3) = Q(3)-LV(3)+P(2)
C
      DO 8 I=1,3
      LAS(I,R) = Q(I)
      RESID(I,R) = FLOAT(LAS(I,R)-MEAN(I))/SQRT(FLOAT(QZERO))
      SUMX(I) = SUMX(I)+Q(I)
      SUMXR(I) = SUMXR(I)+RESID(I,R)
      XSQ(I) = XSQ(I)+Q(I)**2
      RSQ(I) = RSQ(I)+RESID(I,R)**2
    8 CONTINUE
C
      WRITE (6,9) LAS(1,R),RESID(1,R),LAS(2,R),RESID(2,R),LAS(3,R),RESID
     1(3,R)
    9 FORMAT (10X,I15,F15.4,I15,F15.4,I15,F15.4)
   10 CONTINUE
C
C
      DO 13 I=1,3
C
      DO 12 J=1,3
      EXP(I) = FLOAT(SUMX(I))/FLOAT(RUN)
      EXP(J) = FLOAT(SUMX(J))/FLOAT(RUN)
      IF (I.GT.J) COV(I,J)=COV(J,I)
      IF (I.GT.J) GO TO 12
      IF (I.EQ.J) COV(I,J)=(FLOAT(XSQ(I))-2.0*EXP(I)*FLOAT(SUMX(I))+FLOA
     1T(RUN)*(EXP(I)**2.0))/FLOAT(RUN-1)
      IF (I.EQ.J) GO TO 12
      COVSUM = 0.0
C
      DO 11 R=1,RUN
      COVSUM = COVSUM+(FLOAT(LAS(I,R))-EXP(I))*(FLOAT(LAS(J,R))-EXP(J))
   11 CONTINUE
C
      COV(I,J) = COVSUM/FLOAT(RUN-2)
   12 CONTINUE
C
   13 CONTINUE
C
      WRITE (6,14) EXP(1),EXP(2),EXP(3)
```

C.1 BINOMIAL SIMULATION PROGRAM (PROGRAM BIMOD)

```
C
C
C
C
   14 FORMAT (//,10X,3F10.4)
C
      DO 16 I=1,3
      WRITE (6,15) COV(I,1),COV(I,2),COV(I,3)
   15 FORMAT (//,10X,3F20.4)
   16 CONTINUE
C
C
      DO 17 R=1,RUN
      R1(R) = RESID(1,R)*1000.0
      R2(R) = RESID(2,R)*1000.0
      R3(R) = RESID(3,R)*1000.0
   17 CONTINUE
C
C
      DO 20 I=1,3
C
      DO 19 J=1,3
      EXPR(I) = SUMXR(I)/FLOAT(RUN)
      EXPR(J) = SUMXR(J)/FLOAT(RUN)
      IF (I.GT.J) COV(I,J)=COV(J,I)
      IF (I.GT.J) GO TO 19
      IF (I.EQ.J) COV(I,J)=(RSQ(I)-2.0*EXPR(I)*SUMXR(I)+FLOAT(RUN)*(EXPR
     1(I)**2.0))/FLOAT(RUN-1)
      IF (I.EQ.J) GO TO 19
      COVSUM = 0.0
C
      DO 18 R=1,RUN
      COVSUM = COVSUM+(RESID(I,R)-EXPR(I))*(RESID(J,R)-EXPR(J))
   18 CONTINUE
C
      COV(I,J) = COVSUM/FLOAT(RUN-2)
   19 CONTINUE
C
   20 CONTINUE
C
      WRITE (6,14) EXPR(1),EXPR(2),EXPR(3)
C
      DO 21 I=1,3
      WRITE (6,15) COV(I,1),COV(I,2),COV(I,3)
   21 CONTINUE
C
      SCALE(1) = -1233.288
      SCALE(2) = 1217.476
```

```
C                 C.1  BINOMIAL SIMULATION PROGRAM (PROGRAM BIMCC)
C
C
C
      CALL FIX (SCALE)
      CALL HISTG (R1,RUN,32)
      SCALE(1) = -2371.708
      SCALE(2) = 2181.970
      CALL FIX (SCALE)
      CALL HISTG (R2,RUN,32)
      SCALE(1) = -1849.932
      SCALE(2) = 1660.195
      CALL FIX (SCALE)
      CALL HISTG (R3,RUN,32)
      STOP
      END
```

```
C             C.2  DIFFUSION APPROXIMATION MODEL (PROGRAM MNPWR)
C
C
C
      REAL M
      INTEGER TIME,T,TL1,STATES
      WRITE (6,1)
    1 FORMAT (10X,'INPUT STATES--I2, AND TIME--I2',//)
      READ (5,2) STATES,TL1
      TIME = TL1+1
    2 FORMAT (2I2)
      DIMENSION M(6,6,2), CFLOT(6,30), F(6,30), Q(6,2), VAR(6,6), U(6),
     1V(6), EXP(6), Z(6), LOW(6), LARGE(6), PL(6,2), DL(6,2), DQ(6,2)
      L(6) = 0.0
      SUM1 = 0.0
      SUM2 = 0.0
      KK = 0
C
      DO 3 I=1,STATES
      Q(1,I) = 0.0
      Q(I,2) = 0.0
    3 CONTINUE
C
      WRITE (6,4)
    4 FORMAT (10X,'ENTER VALUES FOR THE INITIAL AND DESIRED FORCE LEVELS
     1 USING 2F10.0',//)
C
      DO 6 I=1,STATES
      READ (5,5) PL(I,1),DL(I,1)
    5 FORMAT (2F10.0)
      SUM1 = SUM1+PL(I,1)
      SUM2 = SUM2+DL(I,1)
    6 CONTINUE
C
      QZERO = SUM1
C
      DO 7 I=1,STATES
      Q(I,1) = PL(I,1)/SUM1
      DQ(I,1) = DL(I,1)/SUM2
    7 CONTINUE
C
C
      DO 9 I=1,STATES
C
      DO 8 J=1,STATES
      M(I,J,1) = 0.0
      M(I,J,2) = 0.0
      VAR(I,J) = 0.0
```

```
C                C.2  DIFFUSION APPROXIMATION MODEL (PROGRAM MNPWR)
C
C
C
      L(I) = 0.0
      V(I) = 0.0
      IF (I.EQ.J) M(I,J,1)=FL(I,1)/QZERO
    8 CONTINUE
C
    9 CONTINUE
C
C
      DO 57 T=2,TIME
      TL1 = T-1
      IF (T.NE.2.AND.KK.EQ.2) GO TO 23
      IF (T.NE.1.AND.KK.EQ.1) GO TO 15
      WRITE (6,10)
   10 FORMAT (//,10X,'ENTER ATTRITION RATE FOR EACH STATE',//)
C
      DO 12 I=1,STATES
      READ (5,11) V(I)
   11 FORMAT (F10.0)
   12 CONTINUE
C
      IF (KK.EQ.0) WRITE (6,13)
   13 FORMAT (//,10X,'DO YOU WANT THE CAPABILITY TO CHANGE PROMOTION RAT
     1ES AT EACH TIME PERIOD?',/,10X,'ENTER 1 IF YES, OTHERWISE ENTER 2'
     2,//)
      IF (KK.EQ.0) READ (5,14) KK
   14 FORMAT (I1)
      IF (KK.EQ.2) GO TO 21
   15 WRITE (6,16) TL1
   16 FORMAT (//,10X,'ENTER NEW PROMOTION RATES FOR TIME = ',I2,//)
C
      DO 18 I=1,STATES
      READ (5,17) U(I)
   17 FORMAT (F5.0)
   18 CONTINUE
C
      WRITE (6,19)
   19 FORMAT (//,10X,'ENTER THE NEW RECRUITMENT RATE',//)
      READ (5,20) A
   20 FORMAT (F5.0)
      GO TO 30
   21 LESS1 = STATES-1
C
      DO 22 I=1,LESS1
      J = STATES-I
```

C.2 DIFFUSION APPROXIMATION MODEL (PROGRAM MNPWR)

```
C
C
C
C
      JP1 = J+1
      U(J) = (DQ(JP1,1)/DQ(J,1))*(U(JP1)+V(JP1))
   22 CONTINUE
C
      A = CL(1,1)*(U(1)+V(1))/QZERO
      GO TO 30
   23 WRITE (6,24) TL1
   24 FORMAT (/,10X,'DO YOU WISH TO CHANGE THE ATTRITION RATES FOR TIME
     1= ',I2,/,10X,'ENTER 1 IF CHANGE DESIRED, 2 IF NO CHANGE DESIRED',/
     2/)
      READ (5,25) K
   25 FORMAT (I1)
      IF (K.EQ.2) GO TO 30
      WRITE (6,26)
   26 FORMAT (/,10X,'ENTER THE NEW ATTRITION RATES FOR EACH CLASS----F10
     1.0',//)
C
      DO 28 I=1,STATES
      READ (5,27) V(I)
   27 FORMAT (F10.0)
   28 CONTINUE
C
C
      DO 29 I=1,LESS1
      J = STATES-1
      JP1 = J+1
      U(J) = (DQ(JP1,1)/DQ(J,1))*(U(JP1)+V(JP1))
   29 CONTINUE
C
      A = CL(1,1)*(U(1)+V(1))/QZERO
   30 Q(1,2) = A+(1.0-U(1)-V(1))*Q(1,1)
C
      DO 31 I=2,STATES
      IL1 = I-1
      Q(I,2) = U(IL1)*Q(IL1,1)+(1.0-U(I)-V(I))*Q(I,1)
   31 CONTINUE
C
C
      DO 33 I=1,STATES
C
      DO 32 J=1,STATES
      IL1 = I-1
      JL1 = J-1
      IF (I.GT.J) GO TO 32
```

```
C                 C.2  DIFFUSION APPROXIMATION MODEL (PROGRAM MNPWR)
C
C
C
      IF (I.EQ.1.AND.J.EQ.1) M(1,1,2)=(1.0-U(1)-V(1))**2.0*M(1,1,1)+A*(1
     1.0-A)+(U(I)*(1.0-U(I))+V(I)*(1.0-V(I)))*Q(1,1)
      IF (I.EQ.1.AND.J.EQ.1) GO TO 32
      IF (I.EQ.1.AND.J.GT.2) M(1,J,2)=U(JL1)*(1.0-U(1)-V(1))*M(1,JL1,1)+
     1(1.0-U(1)-V(1))*(1.0-U(J)-V(J))*M(1,J,1)
      IF (I.EQ.1.AND.J.GT.2) M(J,1,2)=M(1,J,2)
      IF (I.EQ.1.AND.J.GT.2) GO TO 32
      IF (I.NE.1.AND.J.EQ.I) M(I,I,2)=U(IL1)**2.0*M(IL1,IL1,1)+(1.0-U(I)
     1-V(I))**2.0*M(I,I,1)+U(IL1)*(1.0-U(IL1))*Q(IL1,1)+(U(I)*(1.0-U(I))
     2+V(I)*(1.0-V(I)))*Q(I,1)+2.0*(U(IL1)*(1.0-U(I)-V(I)))*M(IL1,I,1)
      IF (I.NE.1.AND.J.EQ.I) GO TO 32
      IF (I.EQ.1.AND.J.EQ.2) M(1,2,2)=U(1)*(1.0-U(1)-V(1))*M(1,1,1)+(1.0
     1-U(1)-V(1))*(1.0-U(2)-V(2))*M(1,2,1)-U(1)*(1.0-U(1))*Q(1,1)
      IF (I.EQ.1.AND.J.EQ.2) M(2,1,2)=M(1,2,2)
      IF (I.EQ.1.AND.J.EQ.2) GO TO 32
      IP1 = I+1
      IF (I.NE.1.AND.J.EQ.IP1) M(I,J,2)=U(IL1)*U(I)*M(IL1,I,1)+U(IL1)*(1
     1.0-U(J)-V(J))*M(IL1,J,1)+U(I)*(1.0-U(I)-V(I))*M(I,I,1)+(1.0-U(I)-V
     2(I))*(1.0-U(J)-V(J))*M(I,J,1)-U(I)*(1.0-U(I))*Q(I,1)
      IF (I.NE.1.AND.J.EQ.IP1) M(J,I,2)=M(I,J,2)
      IF (I.NE.1.AND.J.EQ.IP1) GO TO 32
      M(I,J,2) = U(IL1)*U(JL1)*M(IL1,JL1,1)+U(IL1)*(1.0-U(J)-V(J))*M(IL1
     1,J,1)+U(JL1)*(1.0-U(I)-V(I))*M(I,JL1,1)+(1.0-U(I)-V(I))*(1.0-U(J)-
     2V(J))*M(I,J,1)
      M(J,I,2) = M(I,J,2)
   32 CONTINUE
C
   33 CONTINUE
C
      WRITE (6,34) TL1
   34 FORMAT (//,10X,'TIME = ',I2,/)
      WRITE (6,35) TL1
   35 FORMAT (10X,'Q-VALUES AT TIME= ',I2,//,17X,'Q1',18X,'Q2'
     1 ,18X,'Q3',18X,'Q4',18X,'Q5',18X,'Q6',//)
      WRITE (6,36) Q(1,2),Q(2,2),Q(3,2),Q(4,2),Q(5,2),Q(6,2)
   36 FORMAT (6F20.4,//)
C
      DO 38 I=1,STATES
C
      DO 37 J=1,STATES
      M(I,J,1) = M(I,J,2)
      Q(I,1) = Q(I,2)
   37 CONTINUE
C
```

```
C              C.2  DIFFUSION APPROXIMATION MODEL (PROGRAM MNPWR)
C
C
C
   38 CONTINUE
C
      WRITE (6,39) TL1
   39 FORMAT (10X,'THE EXPECTED FORCE LEVEL IN EACH RANK AT TIME ',I2,//
     1,22X,'1',19X,'2',19X,'3',19X,'4',19X,'5',19X,'6',//)
C
      DO 40 I=1,STATES
      EXP(I) = QZERO*Q(I,2)
   40 CONTINUE
C
      WRITE (6,41) EXP(1),EXP(2),EXP(3),EXP(4),EXP(5),EXP(6)
   41 FORMAT (5X,6F20.5)
      WRITE (6,42)
   42 FORMAT (//)
      WRITE (6,43) TL1
   43 FORMAT (10X,'THE COVARIANCE MATRIX FOR FORCE LEVELS AT TIME ',I2,/
     1/,22X,'1',19X,'2',19X,'3',19X,'4',19X,'5',19X,'6',//)
C
      DO 45 I=1,STATES
C
      DO 44 J=1,STATES
      VAR(I,J) = M(I,J,2)*QZERO
   44 CONTINUE
C
   45 CONTINUE
C
C
      DO 47 I=1,STATES
      WRITE (6,46) I,VAR(I,1),VAR(I,2),VAR(I,3),VAR(I,4),VAR(I,5),VAR(I,
     16)
   46 FORMAT (3X,I2,6F20.5,//)
   47 CONTINUE
C
      WRITE (6,48)
   48 FORMAT (//,10X,'DO YOU WISH TO COMPUTE PROBABILITIES OF BEING BETW
     1EEN PLUS OR MINUS ONE SIGMA IN EACH RANK?',/,10X,'ENTER 1 IF ANSWE
     2R IS YES, 2 IF ANSWER IS NO',//)
      READ (5,49) K
   49 FORMAT (I1)
      IF (K.EQ.2) GO TO 57
C
      DO 51 I=1,STATES
      LOW(I) = EXP(I)-SQRT(VAR(I,I))
      LARGE(I) = EXP(I)+SQRT(VAR(I,I))
```

```
C                C.2  DIFFUSION APPROXIMATION MODEL (PROGRAM MNPWR)
C
C
C
      ZLOW = (LOW(I)-EXP(I))/SQRT(VAR(I,I))
      ZLARGE = (LARGE(I)-EXP(I))/SQRT(VAR(I,I))
      CALL MDNOR (ZLOW,PLOW)
      CALL MDNOR (ZLARGE,PLARGE)
      P(I,T) = PLARGE-PLOW
      WRITE (6,50) I,LOW(I),LARGE(I),P(I,T)
   50 FORMAT (10X,'THE ESTIMATED PROBABILITY OF BEING ABLE TO MAINTAIN R
     1ANK ',I1,',',/,10X,'BETWEEN RANK LEVEL  ',I4,'  AND RANK LEVEL  ',
     2I4,'  IS ',F10.5,//)
   51 CONTINUE
C
      WRITE (6,52)
   52 FORMAT (//,20X,'IF YOU WISH TO COMPUTE THE PROBABILITY OF BEING AB
     1LE',/,10X,'TO MAINTAIN ANY TWO RANKS BETWEEN PLUS  AND MINUS ONE',
     2/,10X,'STANDARD DEVIATION, ENTER 1, OTHERWISE ENTER 2.',//)
      READ (5,53) K
   53 FORMAT (I1)
      IF (K.NE.1) GO TO 57
C
      DO 56 I=1,STATES
C
      DO 55 J=1,STATES
      IF (I.EQ.J) GO TO 55
      IF (I.GT.J) GO TO 55
      RHO = VAR(I,J)/(SQRT(VAR(I,I))*SQRT(VAR(J,J)))
      BIVAR = 0.0
      ZLARI = (LARGE(I)-EXP(I))/SQRT(VAR(I,I))
      ZLOWI = (LOW(I)-EXP(I))/SQRT(VAR(I,I))
      ZLARJ = (LARGE(J)-EXP(J))/SQRT(VAR(J,J))
      ZLOWJ = (LOW(J)-EXP(J))/SQRT(VAR(J,J))
      CALL MDBNOR (ZLARI,ZLARJ,RHO,PROB,IER)
      BIVAR = PROB
      CALL MDBNOR (ZLOWI,ZLARJ,RHO,PROB,IER)
      BIVAR = BIVAR-PROB
      CALL MDBNOR (ZLARI,ZLOWJ,RHO,PROB,IER)
      BIVAR = BIVAR-PROB
      CALL MDBNOR (ZLOWI,ZLOWJ,RHO,PROB,IER)
      BIVAR = BIVAR+PROB
      WRITE (6,54) I,J,BIVAR
   54 FORMAT (//,20X,'THE BIVARIATE PROBABILITY OF BEING ABLE TO ',/
     1 ,10X,'MAINTAIN RANK ',I2,'  PLUS AND MINUS ONE SIGMA AND',/,
     2 10X,'RANK ',I2,'  PLUS AND MINUS ONE SIGMA = ',F10.4,//)
   55 CONTINUE
C
```

```
C                    C.2  DIFFUSION APPROXIMATION MODEL (PROGRAM MNFWR)
C
C
C
   56 CONTINUE
C
   57 CONTINUE
C
      STOP
      END
```

```
C                C.3  MINIMUM CONVERGENCE TIME PROGRAM (PROGRAM MINI)
C
C
C
      REAL LOWER(6),LOWERA
      INTEGER RUN,TIME(6)
      DIMENSION U(6), V(6), UPPER(6), DQ(6), PQ(6)
      QTOTAL = 0.0
C
      DO 2 I=1,6
      READ (5,1) V(I),UPPER(I),LOWER(I),DQ(I),PQ(I)
    1 FORMAT (3F5.0,2F4.0)
      QTOTAL = QTOTAL+PQ(I)
      TIME(I) = 0
    2 CONTINUE
C
      READ (5,3) UPPERA,LOWERA
    3 FORMAT (2F5.0)
      U(6) = 0.0
C
      DO 9 RUN=1,50
      U(5) = (PQ(6)*V(6)+DQ(6)-PQ(6))/PQ(5)
      IF (U(5).GT.UPPER(5)) U(5)=UPPER(5)
      IF (U(5).LT.LOWER(5)) U(5)=LOWER(5)
C
      DO 4 J=2,5
      I = 6-J
      IP1 = I+1
      U(I) = ((V(IP1)+U(IP1))*PQ(IP1)+DQ(IP1)-PQ(IP1))/PQ(I)
      IF (U(I).GT.UPPER(I)) U(I)=UPPER(I)
      IF (U(I).LT.LOWER(I)) U(I)=LOWER(I)
    4 CONTINUE
C
      A = ((U(1)+V(1))*PQ(1)+DQ(1)-PQ(1))/QTOTAL
      IF (A.GT.UPPERA) A = UPPERA
      IF (A.LT.LOWERA) A = LOWERA
C
      DO 5 J=2,6
      I = 8-J
      IL1 = I-1
      PQ(I) = (1-V(I)-U(I))*PQ(I)+U(IL1)*PQ(IL1)
      IF (ABS(DQ(I)-PQ(I)).LE.0.001.AND.TIME(I).EQ.0) TIME(I)=RUN
    5 CONTINUE
C
      PQ(1) = (1-V(1)-U(1))*PQ(1)+A*QTOTAL
      IF (ABS(DQ(1)-PQ(1)).LE.0.001.AND.TIME(1).EQ.0) TIME(1)=RUN
      WRITE (6,6) U(1),U(2),U(3),U(4),U(5),A
    6 FORMAT (//,14X,'U1',8X,'U2',8X,'U3',8X,'U4',8X,'U5',9X,'A',//,7X,6
```

```
C                C.3  MINIMUM CONVERGENCE TIME PROGRAM (PROGRAM MINI)
C
C
C
C
     1F10.4)
      WRITE (6,7) PQ(1),PQ(2),PQ(3),PQ(4),PQ(5),PQ(6)
    7 FORMAT (//,14X,'Q1',8X,'Q2',8X,'Q3',8X,'Q4',8X,'Q5',8X,'Q6',//,9X,
     16F10.4,//)
C
      DO 8 I=1,6
      IF (TIME(I).EQ.0) GO TO 9
    8 CONTINUE
C
      GO TO 10
    9 CONTINUE
C
C
   10 DO 12 I=1,6
      WRITE (6,11) I,TIME(I)
   11 FORMAT (//,10X,'THE TIME THAT RANK ',I2,' REACHED THE DESIRED LEVE
     1L OCCURED AT TIME = ',I3)
   12 CONTINUE
C
      STOP
      END
```

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